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ON STATISTICS OF LONGITUDINAL NONLINEAR RANDOM WAVES IN AN ELASTIC BODY

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Probability distributions and spectra of the plane longitudinal nonlinear waves in an elastic body, the stresses in which are linearly dependent on the deformations, are computed. It is shown that the three-dimensional spectrum of the strongly nonlinear elastic waves decays, over a certain interval of the wave numbers, according to a power law. Such inertial intervals exist, as we know in the spectra of many nonlinear random waves arising e.g. in a turbulent motion of a fluid. The result obtained in the present paper indicates that a similar inertial interval can also be discovered in the spectra of the nonlinearly interacting elastic waves accompanied by appreciable deformations. Exact expressions for the probability distributions and spectra of the random Riemannian waves were derived in [1-6].

1. Let us consider the plane longitudinal waves in an elastic body obeying Hook's Law. In the Lagrangian formulation the coordinate $x(a, t)$ of the fixed particle of the body with the initial coordinate a satisfies, in this case, the linear equation (see e.g. [7])

$$\frac{\partial^2 x}{\partial t^2} = c^2 \frac{\partial^2 x}{\partial a^2} \tag{1.1}$$

In the Eulerian formulation the corresponding velocity $v(x, t)$ and deformation $J(x, t)$ fields satisfy the equations

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = c^2 J \frac{\partial J}{\partial x} \tag{1.2}$$
$$\frac{\partial J}{\partial t} + v\frac{\partial J}{\partial x} = J\frac{\partial v}{\partial x}$$
$$v(a, t) = \frac{\partial x}{\partial t}, \quad J(a, t) = \frac{\partial x}{\partial a} - 1$$

which, for sufficiently large deformations, are appreciably nonlinear.

The basic aim of the present paper is to determine varying statistical characteristics of the nonlinear random fields $v(x, t)$ and $J(x, t)$, using their known statistical properties prevailing at the initial instant of time. The search for the statistical properties, and in particular for the moments of these fields, by means of direct averaging of Eqs. (1.2), encounters difficulties analogous to those arising in the problem of closure in the theory of turbulence (see e.g. [8]). The linear character of (1.1) however makes it possible to obtain the statistical properties of the random elastic waves in the Lagrangian formulation without much difficulty. It appears that the statistical properties of the waves of diverse physical nature expressed in the Lagrangian and Eulerian formulation are connected by certain very simply universal relations. Using these relations we can determine, with the help of the known statistics of $v(a,t)$ and $J(a,t)$ the unknown statistical properties of the fields $v(x, t)$ and $J(x, t)$. Such a Lagrangian approach is used in the present paper to analyze the statistics of the nonlinear fields $v(x, t)$ and $J(x, t)$.

2. Let us give some of the relations connecting the statistics of the one-dimensional random waves in the Lagrangian and Eulerian formulations, which shall be of use in determining the statistics of the nonlinear elastic waves. We note that similar relations were studied in [9-12] for a turbulent motion of an incompressible fluid. The latter relations however, cease to be applicable when the compressibility effects become appreciable $(J \neq 0)$.

Let the probability density of the Lagrangian fields v, J and x, i.e. $f[v, J, x; a, t]$ be known. Performing the computations analogous to those given in [13] and regarding $x(a, t)$ as a monotonous function of a, we can show that the probability density of the Eulerian fields $v(x, t)$ and $J(x, t)$ which is $w[v, J; x, t]$ is related to f by the following equation:

$$w[v, J; x, t] = (1 + J) \int_{-\infty}^{\infty} f[v, J, x; at]\, da \tag{2.1}$$

For the nonlinear waves in a plasma, in the bundles of charged particles, and in many other cases, the formation of multistream motions is a common characteristic feature. When it happens, $x(a, t)$ becomes a nonmonotonous function of a, and (2.1) is replaced by a more general formula [6, 13]

$$|1 + J| \int_{-\infty}^{\infty} f[v, J, x; a, t]\, da = \sum_{N=1}^{\infty} P(N; x, t) \sum_{n=1}^{N} w[v, J; x, t \mid n, N] \tag{2.2}$$

where $w[v, J; x, t \mid n, N]$ is the Eulerian probability density of the n-th stream under the condition that N streams have formed at the point (x, t). From this it follows that the mean number of streams is

$$\langle N(x, t)\rangle = \sum_{N=1}^{\infty} NP(N; x, t) = \iiint_{-\infty}^{\infty} |1 + J|\, f[v, J, x; a, t]\, da\, dv\, dJ \tag{2.3}$$

When the initial conditions are arbitrary, the solution of (1.1) can also be represented by the nonmonotonous function $x(a, t)$. However, Eqs. (1.1) and (1.2) cease to describe the elastic waves correctly even before $x(a, t)$ becomes nonmonotonous. For this reason, in the present case $\langle N\rangle$ serves as the measure of the validity of the statistical results obtained. We shall assume that the expressions obtained below describe the statistics of the nonlinear elastic waves sufficiently well, as long as $\langle N\rangle$ is nearly unity, this implying that the function $x(a, t)$ is monotonous practically everywhere.

In what follows, we shall assume that the functions $v(x, t)$ and $J(x, t)$

are statistically homogeneous and single-valued in x. The formulas (2.1) and (2.2) will now assume a particularly simple form

$$w[v, J; t] = (1+J) f[v, J; t] \tag{2.4}$$

We shall also give an expression for the three-dimensional spectrum of the Eulerian field $v(x, t)$ statistically homogeneous in x. We shall write it as follows:

$$G[\Omega, t] = \lim_{X\to\infty} \frac{1}{2X} \iint_{-X}^{X} \langle v(x_1, t) v(x_2, t)\rangle \exp[i\Omega(x_1 - x_2)]\, dx_1\, dx_2$$

Assuming $x(a, t)$ to be a monotonous function of a and integrating over the Lagrangian coordinates, we obtain

$$G[\Omega, t] = \lim_{X\to\infty} \frac{1}{2X} \iint_{-A_1}^{A_2} \Big\langle \prod_{j=1}^{2} [1 + J(a, t)]\, v(a_j, t) \times$$

$$\exp\{i\Omega[x(a_1, t) - x(a_2, t)]\}\Big\rangle da_1\, da_2$$

where $A_{1,2}$ are solutions of the equations $\pm X = x(a, t)$. Since $1 + J = \partial x / \partial a$ and on the finite time intervals we have $|X - A| / A \sim 1/X$ as $X \to \infty$, we write the last equation in the form

$$G[\Omega, t] = \frac{1}{\Omega^2} \lim_{A\to\infty} \frac{1}{A_1 + A_2} \iint_{-A_1}^{A_2} \Big\langle \frac{\partial v(a_1, t)}{\partial a_1} \frac{\partial v(a_2, t)}{\partial a_2} \times$$

$$\exp[i\Omega[x(a_1, t) - x(a_2, t]\}\Big\rangle da_1\, da_2$$

The averaged integrand depends, by virtue of the statistical homogeneity, only on $s = a_1 - a_2$, therefore we finally obtain

$$G[\Omega, t] = \frac{1}{\Omega^2} \int_{-\infty}^{\infty} \Big\langle \frac{\partial v(a, t)}{\partial a} \frac{\partial v(a+s, t)}{\partial a} \times \tag{2.5}$$

$$\exp\{i\Omega[x(a, t) - x(a + s, t)]\}\Big\rangle ds$$

3. Let us now analyze the Eulerian statistics of the plane elastic random waves. We shall consider, for definiteness, the case when $J(a, 0) = 0$ and $v(a, 0) = v_0(a)$ is a statistically homogeneous function all probabilistic properties of which are known, we have

$$x(a, t) = a + \frac{1}{2c} \int_{a-ct}^{a+ct} v_0(s)\, ds \tag{3.1}$$

First we find the probabilistic point distribution $w[v, J; t]$ of the fields $v(x, t)$ and $J(x, t)$. In accordance with (2.4) and (3.1), the distribution is connected with the initial two-point distribution $w_0[v_1, v_2; a_1 - a_2]$ of $v_0(a)$ by the formula

$$w[v, J; t] = 2(1+J) w_0[v+J, J-v; 2ct]$$

Let us quote the physical corollaries of this formula, i.e. the expressions for the mean kinetic energy density and mean square of the velocity of the noise elastic waves

$$^1/_2 \langle \rho (x, t) v^2 (x, t) \rangle = $$
$$^1/_4 [\langle v_0^2 (a) \rangle + \langle v_0 (a + ct) v_0 (a - ct) \rangle]$$
$$\langle v^2 (x, t) \rangle = \langle \rho v^2 \rangle +$$
$$^1/_8 c [\langle v_0^2 (a + ct) v_0 (a - ct) \rangle - \langle v_0^2 (a - ct) v_0 (a + ct) \rangle]$$

We assume here, for simplicity, that the body density in the undeformed state is unity everywhere, so that $\rho = 1 / (1 + J)$. For the random function $v_0 (a)$ statistically reversible or symmetrically distributed with respect to $v = 0$, we have $\langle v^2 \rangle = \langle \rho v^2 \rangle$. When $ct \gg l$, where l is the correlation length of $v_0 (a)$, the elastic waves tend to the state of statistical equilibrium in which $\langle v^2 (x, \infty) \rangle = \langle v_0^2 \rangle / 2$.

The analysis of more complex statistical characteristics of the elastic waves, such as their spectra, requires the knowledge of the distribution of $x (a, t)$ and hence, according to (3.1), of the statistics of the linear functional of $v_0 (a)$. This problem has a simple solution if $v_0 (a)$ is a Gaussian random function. However, the functions $v (x, t)$ and $J (x, t)$ satisfying Eqs. (1.2) become, with the finite probability $| v_0 | > c$, nonunique functions of x and no longer describe correctly the behavior of the elastic waves. Nevertheless, if the mean number of their flows $\langle N \rangle$ is close to unity, the function $v (x, t)$ is singlevalued for practically every value of x, and the elastic waves are described by Eqs. (1.1) and (1.2) sufficiently well.

Let us obtain $\langle N \rangle$ for the Gaussian $v_0 (a)$ with the zero mean value and correlation function $K [s]$. From (2.3) it follows that

$$\langle N (x, t) \rangle = \sqrt{\frac{2}{\pi}} \int_0^{1/\sqrt{D}} \exp \left\{ - \frac{z^2}{2} \right\} dz + \sqrt{\frac{D}{\pi}} \exp \left\{ - \frac{1}{D} \right\}$$

$$D (t) = [K [0] - K [2ct]] / c^2$$

Even for $D = 2$ when the nonlinearity of Eq. (1.2) is appreciable and the fluctuations of J are of the order of unity, $\langle N \rangle \approx 1.17$, i.e. the value is sufficiently near to unity. Therefore, we shall assume in the course of analysing the spectrum of $v (x, t)$, that $v_0 (a)$ is Gaussian ($\langle v_0^2 \rangle \leqslant 2c^2$).

4. Let us analyse the three-dimensional spectrum of the Eulerian velocity field, of the plane elastic waves

$$G [\Omega; t] = \int_{-\infty}^{\infty} \langle v (x, t) v (x + s, t) \rangle \exp \{ i \Omega s \} ds$$

In order to avoid cumbersome manipulations, we shall investigate in detail only the equilibrium case of $t \to \infty$. Assuming $v_0 (a)$ to be Gaussian with the correlation function $K [s]$ we have, in accordance with (2.5),

$$G [\Omega, t] = - \frac{1}{2\Omega^2} \int_{-\infty}^{\infty} \frac{d^2 K [s]}{ds} \exp \{ i \Omega s - B (s) \Omega^2 \} ds + \frac{1}{2} \cos 2c\Omega t \exp \left\{ - \Omega^2 \frac{t}{c} \int_0^{\infty} K [s] ds \right\} \int_{-\infty}^{\infty} \left\{ K [s] - \frac{1}{2c^2} K^2 [s] \right\} ds \quad B (s) = \frac{1}{2c^2} \int_0^s (s - a) K [a] da \left(\int_0^{\infty} K [s] ds \neq 0 \right). \qquad (4.1)$$

The first term of the above expression describes the spectrum of the waves moving in the same direction, and the second term describes the combined spectrum of the waves moving in different directions. At sufficiently large t the latter is concentrated in a narrow range Ω which decreases with increasing t. This is due to the fact that even in the case of waves which are practically linear ($K[0] \ll c^2$) the Eulerian distance between the correlated values of the waves moving in various directions differs from the Lagrangian distance $2ct$ by a random quantity which is much greater than l.

Let us consider the spectrum of the waves moving in the same direction

$$G[\Omega] = -\frac{1}{2\Omega^2} \int_{-\infty}^{\infty} \frac{d^2K[s]}{ds^2} \exp\{i\Omega s - B(s)\Omega^2\} ds \qquad (4.2)$$

Its Fourier transform is $\Pi(s) \langle v(x, t)\, v(x + st)\rangle$, where

$$\Pi(s) = \begin{cases} 1, & l \ll |s| \ll ct \\ 0, & |s| \gtrsim ct \end{cases}$$

is a function truncating the correlation when $s \sim 2ct$. The corresponding equilibrium spectrum for the case when the nonlinearity of (1.2), i.e. the difference between the Lagrangian and Eulerian coordinates can be neglected, has the form

$$G[\Omega] = \frac{1}{2} \int_{-\infty}^{\infty} K[s]\, e^{i\Omega s}\, ds$$

The above spectrum is obtained by neglecting in (4.2) the term $B(s)\Omega^2$ describing the difference between the Lagrangian and Eulerian coordinates. Expanding the right-hand side of (4.2) into a series in powers of $B(s)\Omega^2$, we obtain an expression for the equilibrium velocity spectrum in powers of the nonlinear interactions. We can limit ourselves in such an expansion to the first few terms, provided that $K[0] \ll c^2$ or $B(l)\Omega^2 \ll 1$. Assuming in this expansion $\Omega = 0$, we obtain the exact value of the spectrum at zero

$$G[0] = \frac{1}{2} \int_{-\infty}^{\infty} K[s]\, ds + \frac{1}{4c^2} \int_{-\infty}^{\infty} K^2[s]\, ds$$

We note that the value at zero is retained for the complete spectrum (4.1).

Above we have obtained the equilibrium dispersion of the velocity $\langle v^2(x, \infty)\rangle = K[0] / 2$. Using this together with $G[0]$, we can determine the effective correlation length of the waves moving in the same direction for $ct \gg l$

$$l_\infty = \frac{G[0]}{\langle v^2(x, \infty)\rangle} = l + \frac{1}{2c^2 K[0]} \int_{-\infty}^{\infty} K^2[s]\, ds$$

where l denotes the correlation length of $v_0(a)$. Thus the nonlinearity leads to a reduction in the effective width $\Omega_\infty = 1 / l_\infty$ of the spectrum. We note that the width of the spectrum of the Riemannian wave remains unchangeable [4].

When Ω $(B(l)\Omega^2 \gg 1)$ are large, the spectrum (4.2) decays according to the universal power law. Using the saddle point method to compute the integral in

(4.1), we obtain

$$G[\Omega]_{\Omega\to\infty} = \frac{cH}{\Omega^3}\sqrt{\frac{\pi}{K[0]}}\exp\left\{-\frac{c^2}{K[0]}\right\}, \quad H = \frac{\partial^2 K}{\partial s^2}\Big|_{s=0}$$

A similar asymptotics was obtained for the spectrum of the Riemannian waves in [1,4,14]. In [4] it was noted that the law $G \sim 1/\Omega^3$ is intimately connected with the appearance of multistreaming. The power law spectrum however is formed even before the multistreaming appears and is inherent in any strongly nonlinear perturbation. To show this, we consider the spectrum in the case when $v_0(a) = A\sin(pa + \varphi)$ where φ denotes the random phase uniformly distributed over the interval $[0, 2\pi]$. When $A \lesssim c$, the waves in a rod will be nonlinear, but single-stream everywhere. In addition, as we have shown above, the spectrum has an inertial region in which it decays according to a power law.

Performing the computations we obtain from (2.5)

$$G[\Omega, t] = \frac{\pi}{2}A^2\cos^2 pct \sum_{n=-\infty}^{\infty}\frac{b(n,t)}{n^2}\delta(\Omega - pn)$$

$$b(n, t) = J_{n+1}^2(nz) + J_{n-1}^2(nz) + 2J_{n+2}(nz)\,J_{n-2}(nz)$$
$$z = (A/c)\sin pct$$

In the region $1 \ll n \ll 1/(1-z)$ we have $b(n, t) \sim 1/\sqrt{n}$ [15], so that $G[n, t] \sim n^{-5/2}$ decays according to a power law. When $n(1-z) \gtrsim 1$, the function $b(n, t)$ decays exponentially.

In conclusion we give a case in which the integral in (4.2) can be computed exactly. Let

$$K[s] = \begin{cases}(k/h)(h - |s|), & |s| \leqslant h\\ 0, & |s| > h\end{cases}$$

Then

$$G[\Omega] = \frac{k}{h\Omega^2}\left[1 - \exp\left(-\frac{\Omega^2h^2k^2}{6c^2}\right)\cos\Omega h\right]$$

Here $G[\Omega]$ also decays according to a power law $G \sim 1/\Omega^2$ when $\Omega \to \infty$, but in this case the law does not depend on the nonlinearity of (1.2) but on the nondifferentiability of the initial velocity $v_0(a)$. When $k \to 0$, the spectrum $G[\Omega]$ transforms into an equilibrium spectrum of linearized equations.

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